SLE(κ, ρ)and Boundary Coulomb Gas

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Abstract

We consider the coulomb gas model on the upper half plane with different boundary conditions, namely Drichlet, Neuman and mixed. We related this model to SLE(κ, ρ) theories. We derive a set of conditions connecting the total charge of the coulomb gas, the boundary charges, the parameters κ and ρ. Also we study a free fermion theory in presence of a boundary and show with the same methods that it would lead to logarithmic boundary changing operators.

Keywords: conformal field theory, SLE equation

1 Introduction

Conformal field theories have found many applications in classification of phase transitions and critical phenomena in two dimensions and other areas such as string theory. In particular the minimal models introduced in [1] reveal many exact solutions to various two dimensional phase transitions like Ising model at critical point or three critical Ising model and so on. These models were first considered on the whole plane, but as many surface phenomena are very interesting to analyze, boundary conformal field theory was soon developed [2]. Essentially it was shown that conformal field theory in the half plane with proper boundary condition (BC) could be mapped to a whole-plane conformal field theory, the price you have to pay is to insert image fields in the other half plane. The idea was then applied to different problems such as turbulence [3].

On the other hand, recently a new method to investigate the so called geometrical phase transitions has been developed, which were previously described by conformal field theories. The new method, Stochastic Lowener Evolution (SLE) [4] is a probabilistic approach to study scaling behavior of geometrical models. For a review see [5, 6, 7, 8]. SLE's can be simply stated as conformally covariant processes, defined on the upper half plane, which describe the evolution of random domains, called SLE hulls. These random domains represent critical clusters.

The idea of SLE was first developed by Schramm [4]. He showed that under assumption of conformal invariance, the scaling limit of loop-erased random walks is SLE_2. (SLE's are

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parameterized by a real number k, and are abbreviated as SLE_k) Also he claimed without proof that SLE_6 is the scaling limit of critical percolation. The claim was proved later on by Smirnov [9]. He showed that in the scaling limit of percolation, conformal invariance exists and also using the new technic proved Cardy's formula [10, 11].

As in SLE we are dealing with conformal mappings and geometrical phase transitions, one expects that there should be a relation between SLE and conformal field theory. The existence of a boundary is an essential point in SLE, so one would expect it would be related to BCFT's. An explicit relationship between the two was discovered by Bauer and Bernard [13, 14, 15, 16] and afterwards by [17, 18, 19]. They coupled SLE_κ to boundary conformal field theory with specific central charge depending on the parameter κ. These CFT's live on the complement of SLE hulls in the upper half plane In such situations, boundary states emerge on the boundary conformal field theory. The good point is that these states are zero modes of the SLE_κ evolution, that is they are conserved in mean. This means that all components of these states are local martingales of SLE_κ and hence one is able to compute crossing probabilities in purely algebraic terms. The key point of these result is existence of null vectors, just an in the case of minimal models. It turns out that the Verma module in the CFT part should contain level 2 null state, though higher level null vectors have been considered, too [20, 21]. The level two null state ,$|h\rangle$, obeys the equation

$$\left(2L_{-2} + \frac{\kappa}{2}L_{-1}^2\right)|h\rangle = 0, \tag{1}$$

where L_{-k}'s are the Virasoro operatores. This equation naturally leads to second order differential equations on correlators containing the null state which are the same ones obtained from stochastic approach.

More recently a generalization of these theories has emerged: the so called $\mathrm{SLE}(\kappa, \rho)$ [28, 29]. The new theory describes random growing interfaces in a planar domain which have markovian property and conformal invariancy. In fact $\mathrm{SLE}(\kappa, \rho)$ is the minimal way to generalize the original SLE while keeping self-similarity. We will come back to the properties of such theories in the third section.

Cardy [27] extended the correspondence between SLE_κ and CFT, to the case of $\mathrm{SLE}(\kappa, \rho)$. In this case the equation in the CFT part is a little bit different from equation (1), it has an extra term which comes from a current density $\mathcal{J}$:

$$\left(2L_{-2} + \frac{\kappa}{2}L_{-1}^2 - \mathcal{J}_{-1}L_{-1}\right)|h\rangle = 0. \tag{2}$$

If a current density is already present in the theory, then the equation would have a more physical meaning. The example considered in [27] is a free bosonic field with the action $(g/4\pi)\int(\partial\phi)^2 d^2z$. The boundary conditions are piecewise constant Drichlet; that is on real axis we have

$$\phi(x) = 2\pi \sum_j \alpha_j H(x - x_j) \tag{3}$$

where $H(x)$ is the Heavyside step function.

The effect of a jump of strength α could be replaced with a boundary condition changing operator ϕ_α. The state produced by operation of this field is the one which satisfies the equation (2). The correlation functions associated with such field can be defined in the following way: first find a classical solution satisfying the boundary condition (BC). Calling the solution ϕ_{cl}, we write $\phi = \phi_{cl} + \phi'$, where ϕ' is assumed to vanish on the boundary. The partition function is then calculated to be $Z = Z_\alpha Z'$. Note that this happens, because

the theory is free. In boundary conformal field theory (BCFT) one can think of Z_α as the correlation function $\langle\prod_j \phi_{\alpha_j}\rangle$.

Let's turn back to the specific problem. The classical solution of equation of motion satisfying the boundary condition (3 is

$$\phi_{cl} = i\sum \alpha_j \ln((z-x_j)/(\bar{z}-x_j)). \tag{4}$$

Hence the partition function and the correlators are obtained to be

$$Z_\alpha = \left\langle \prod_j \phi_{\alpha_j} \right\rangle = \prod_{j<k} \left(\frac{x_k - x_j}{a}\right)^{2g\alpha_j\alpha_k}. \tag{5}$$

In a similar way the expectation value of energy-momentum density in presence of such BC is found, which can be thought of the correlation of energy-momentum tensor with the boundary fields ϕ_α's. Conformal Ward identity comes afterward with which Cardy has found the action L_0, L_{-2} and L_{-1}^2 on the boundary fields. The fields are found to be primary with scaling dimension is $h_i = g\alpha_i^2$. Going back to the correlation functions (5), one finds that the two results are compatible if one assumes that the sum of all jump vanishes.

To connect all these to SLE, one should look if it is possible to satisfy either equation (1) or (2). The first one is satisfied if we assume $\kappa = 4$ and $\alpha_i^2 = \alpha^{*\,2} = 1/4g$, so that $h_i = 1/4$. The other one which is related to SLE(κ,ρ) appears if we loose the second condition, which leads to

$$2L_{-2}\phi_{\alpha_i} = 2L_{-1}^2\phi_{\alpha_i} - \sum_j{}' \frac{\rho_j}{x_j - x_i} L_{-1}\phi_{\alpha_i}, \tag{6}$$

where $\rho_j = (\alpha_j - \alpha_i)(1-(\alpha_i/\alpha^*)^2)$ and prime means that j is excluded from the summation. This equation can be written as (2) in terms of $J = -2ig^{1/2}\partial\phi$ which is conserved due to equations of motion.

Cardy then mentions that it is possible to take Neuman boundary condition (NBC) instead of Drichlet boundary condition (DBC), one just have to consider the dual description of the problem with NBC on dual fields $\tilde{\phi}$ with insertions of vertex operators $e^{i\alpha\tilde{\phi}}$.

In the next section, we will consider the coulomb gas action with a charge at infinity and investigate different boundary conditions. The next chapter is devoted to the connections to SLE and SLE(κ,ρ). In the last section we will apply the same method to the fermionic action of $c=-2$ and find logarithmic operators and some structures which has been found before [25].

2 Boundary Operators in Coulomb Gas Model

Coulomb gas is defined via the following action:

$$S = \frac{1}{4\pi}\int \left[g(\partial\varphi)^2 + 2\,Q\,R\varphi\right] \tag{7}$$

where R is the scaler curvature associated with the background metric. We are interested in the case where φ is defined on the upper half plane. The boundary conditions can have several forms: Dirichlet (DBC), Neuman (NBC) or mixed boundary condition, but not all of them are conformally invariant. A boundary condition is conformally invariant if the antiholomorphic part of the energy momentum tensor in the lower half plane is the analytic continuation of its holomorphic part in the upper half plane. This is satisfied if $T_{xy} = 0$ at $y = 0$ where

$$T_{xy} = \frac{g}{2}\partial_x\varphi\partial_y\varphi - \frac{Q}{2}\partial_x\partial_y\varphi. \tag{8}$$

In NBC case, where $\partial_y \varphi|_{y=0} = 0$, the above condition is automatically satisfied, while in DBC case, it is not satisfied unless if $Q = 0$. Let's take Neuman boundary condition, we construct our desired boundary condition in the following way: suppose we have some electrical charges at points x_j located at the boundary. We would like to find the boundary changing operators associated with these electric charges. The classical solution which satisfies Laplace equation together with the above property has the form

$$\varphi_{cl} = \sum \frac{\lambda_j}{g} \ln(z - x_j)(\bar{z} - x_j) \tag{9}$$

In this solution the normal derivative of field vanishes and hence the solution satisfies NBC. For consistency, φ has to behave like $-2\,Q\ln(z\bar{z})$ when $z, \bar{z} \to \infty$, so one should assume the condition $\sum \lambda_j = -2Qg$.

the next step is to calculate the partition function. If we write $\varphi = \varphi_{cl} + \varphi'$, with the condition $\partial_y \varphi'|_{y=0} = 0$, we'll have $S[\varphi] = S[\varphi_{cl}] + S[\varphi']$, and the partition function associated with $S[\varphi_{cl}]$ is derived to be

$$Z_{cl} = \prod_{j<k} \left(\frac{x_k - x_j}{a} \right)^{\frac{2\lambda_j \lambda_k}{g}}. \tag{10}$$

Here a is the UV cut off. Let Z' be the partition function associated with $S[\varphi']$, then we define the boundary correlation function to be

$$\langle \prod_j \phi_{\lambda_j}(x_j) \rangle = \frac{Z}{Z'} \tag{11}$$

Also any expectation value $\langle O \rangle$ in presence of this boundary condition can be defined in the following way:

$$\langle O \rangle = \frac{\langle O \prod_j \phi_{\lambda_j}(x_j) \rangle}{\langle \prod_j \phi_{\lambda_j}(x_j) \rangle} \tag{12}$$

In particular the expectation value of energy-momentum tensor T can be computed to give:

$$\langle T \rangle = -g(\partial\varphi_{cl})^2 + 2Q\partial^2\varphi_{cl} = \frac{1}{g} \sum_{j,k} \frac{\lambda_j \lambda_k}{(z - x_j)(z - x_k)} + 2\frac{Q}{g} \sum_j \frac{\lambda_j}{(z - x_j)^2}. \tag{13}$$

Taking the limit of $z \to x_i$ one can obtain the conformal weights of the operators $\varphi(x_i)$, $h_{\lambda_i} = \lambda_i(\lambda_i + 2Q)/g$. The result suggests that these operators are related to vertex operators $\exp(i\lambda\phi)$ in the coulomb gas model.

Now let's investigate the Dirichlet boundary condition with $Q = 0$. The field is taken to be piecewise constant with some jumps $2\pi\alpha$ at the points x_j's. In this case, the classical solution of the equation of motion is

$$\varphi_{cl} = i \sum \alpha_j \ln\left(\frac{z - x_j}{\bar{z} - x_j}\right) \tag{14}$$

Note that in the presence of external charge, second term of offdiagonal energy-momentum tensor (8) does not vanish at the points x_j.

At this level, there are no conditions on α_j's, and Z_{cl} can be obtained as in the case of NBC. The result is just the same as equation (10), you just need to transform $g \to 1/g$ and $\lambda_j \to \alpha_j$. Similar to the NBC case, there appear some operators on the points x_j. Also the definition of correlation functions of such fields is the same as equation (11). It can be shown that these operators are related to magnetic vertices.

Again one can define the expectation values of any operator via a relation similar to equation (12). The related fields are then found to be primary ones with conformal weight $h_{\alpha_j} = g\alpha_j^2$. Now if we look back to the correlation functions of φ_{α_i}'s, one observes that they are consistent with the weight derived, if the sum of all jumps add up to zero, that is the value of φ on both far ends of the boundary should be the same[27].

The other BC is the mixed one (with $Q = 0$). In this case φ_{cl} has the following form:

$$\varphi_{cl} = \sum \frac{\lambda_j}{g} \ln(\bar{z} - x_j)(z - x_j) + i \sum \alpha_j \ln \left(\frac{z - y_j}{\bar{z} - y_j} \right) \tag{15}$$

The above BC means that some electric and magnetic charges are fixed on points x_j and y_j. It turns out that in the classical action, the two parts related to magnetic and electric charges decouple. The results are a combination of the previous two cases. If $x_j = y_j$ there is a electric-magnetic vertex operator $O_{e,m}$ with the scaling dimension $h = (\sqrt{g}\alpha + \lambda/\sqrt{g})^2$. It can be argued that for consistency we must have $\sum \lambda_j \alpha_j = 0$ in addition to previous conditions.

As in SLE, one can use similar bulk operators, rather than the boundary ones, in brief we will talk about bulk correlation functions. In the case of DBC one is able to use methods similar to the image method in ordinary electromagnetic theory, so if there is a magnetic charge at z_j then another magnetic charge with opposite sign should be located at z_j^*. Thus the neutrality condition is satisfied automatically. But in Neuman case, the image charge has the same sign and then the neutrality condition is not satisfied automatically. The two point functions of Dirichlet and Neuman BC's have the following form

$$\langle O_e O_{-e} \rangle = \left(\frac{\mathrm{Im}z \ \mathrm{Im}w}{|z - w|^2 |z - w^*|^2} \right)^{e^2} \tag{16}$$

$$\langle O_{m_1} O_{m_2} \rangle = \frac{1}{(\mathrm{Im}z)^{m_1^2} (\mathrm{Im}w)^{m_2^2}} \left| \frac{z - w}{z - w^*} \right|^{2m_1 m_2} \tag{17}$$

These correlation functions could be used to develop a SLE(κ, ρ) theory.

3 Relation to SLE(κ, ρ)

In this section we would like to relate the result derived in the previous section to the new and interesting topic SLE(κ, ρ). As we mentioned in introduction, SLE(κ, ρ) is somehow a more general case of SLE$_\kappa$ which describes random growing interfaces in a planar domain which have markovian property and conformal invariancy. This evolution has the following form

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \qquad g_0(z) = z, \qquad dW_t = \sqrt{\kappa} dB_t + \sum_1^n \frac{\rho_j}{W_t - g_t(x_j)}, \tag{18}$$

where x_j's are some arbitrary points not necessarily on the boundary, and $g_t(z)$ is a conformal map from a subset of upper half plane, H_t, to the whole upper half plane, H. In other words, $H_t = H \backslash K \to H$ where K_t is the complement of H_t in the upper half plane. The subset K_t which is called the hull of evolution, is just the growing domains mentioned above. Setting all of the parameters ρ_j to zero, one arrives at the ordinary SLE$_\kappa$, but with nonzero ρ_j we may have several random curves growing from the points on the boundary [30]. SLE(κ, ρ) has many interesting properties as you can see in [28, 29].

As we said earlier, Cardy [27] has established a connection between SLE(κ, ρ) and a CFT with Drichlet boundary changing operators inserted at points x_j. He derives the properties of the boundary operators ϕ_j which leads to equation (6) which is a specific case of the more general equation

$$\left(2L_{-2} - \frac{\kappa}{2}L_{-1}^2 + \sum \frac{\rho_j L_{-1}}{g_t(x_j)}\right)\phi_j = 0 \tag{19}$$

needed to establish a connection to SLE(κ, ρ).

Let's see how the different choices for boundary conditions in the previous section affect the derived SLE(κ, ρ). Taking $g = 1$, for the case of NBC, calculation of $2L_{-2} - \frac{\kappa}{2}L_{-1}^2$ yields

$$2L_{-2}\phi_j - \frac{\kappa}{2}L_{-1}^2\phi_j = \sum \frac{4\lambda_i(Q-\lambda_j) + \kappa\lambda_i\lambda_j}{(x_i - x_j)^2}\phi_j + (2 - 2\kappa\lambda_i^2){\sum_{j,k}}' \frac{\lambda_j\lambda_k}{(x_i - x_j)(x_i - x_k)}\phi_j \tag{20}$$

If we impose the two conditions $\lambda_i^2 = \dfrac{1}{\kappa} = \lambda^2$ and $Q = \dfrac{\lambda(-\kappa+4)}{4}$, then $2L_{-2} - \frac{\kappa}{2}L_{-1}^2$ will be zero. Also we will have

$$c = \frac{(6-\kappa)(3\kappa-8)}{2\kappa}, \qquad h = \frac{6-\kappa}{2\kappa} \tag{21}$$

which is similar to the case of SLE$_\kappa$.

On the other hand, if we impose only the condition $Q = \dfrac{\lambda(-\kappa+4)}{4}$ we will have

$$2L_{-2}\phi - \frac{\kappa}{2}L_{-1}^2\phi = (2 - 2\kappa\lambda_i^2){\sum_{j,k}}' \frac{\lambda_j\lambda_k}{(x_i - x_j)}L_{-1}\phi \tag{22}$$

which is the equation in SLE(κ, ρ).

There is an equivalent to study this problem. Bauer, Bernard and Kytola (BBK) have shown that SLE(κ, ρ) could be obtained starting from correlation functions of boundary fields [30]. Later Kytola has applied this method to the coulomb gas model, that is he has considered the boundary correlation functions of a coulomb gas and derived the corresponding SLE(κ, ρ)[31]. This method, although is a little bit different from the cardy's one, has the same consequences. By means of our results in the previous section, and using the method of BBK, one is able to derive the corresponding SLE(κ, ρ). Namely the correlations (10) and the one obtained in other BC's are very suitable to do this. Additionally, Schramm and Wilson [32] and also Friedrich and Bauer[33], argued that it not necessary to take the operators just on the boundary and one is able to construct SLE($\kappa, \rho_{bound.}, \rho_{bulk}$) using both the operators on the boundary and in the bulk. To do this, the correlation functions (16) are helpful. The result are very similar to the ones we have derived, you should just replace x_j with z_j where z_j's are points in the upper half plane.

Another point to mention is that coulomb gas action is not the only one to produce these results. Zamolodchikov and Zamolodchikov [34] have considered the Liouville action

$$S_L = \int \left(\frac{1}{4\pi}(\partial\phi)^2 + \mu e^{2b\phi}\right) \tag{23}$$

with the boundary conditions

$$\phi(z,\bar{z}) = -2\log z\bar{z} + O(1) \qquad |z| \to \infty \tag{24}$$

$$\phi(z,\bar{z}) = -2\eta_i \log(z - x_i)(\bar{z} - x_i) + O(1) \qquad |z| \to x_i \tag{25}$$

and have calculated the classical action to be

$$S_{L-cl} = \sum_{i<j} 2\eta_i\eta_j \log|x_i - x_j|^2 \tag{26}$$

where the condition $\sum \eta_i = 1$ should be imposed. As you see this is very similar to the correlation functions in the case of coulomb gas and will lead to the same SLE(κ, ρ).

Also Fateev, Zamolodchikov and Zamolodchikov [35] have considered Liouville action on the disc with proper boundary conditions and derived the correlation functions of the operators in the bulk and on the boundary. Hence it should be related to a radial SLE(κ, ρ).

4 Fermionic Boundary Changing Operators and LCFT's

In this section we will apply the same method not to the action of a bosonic field, but to the action of a free fermionic fields:

$$S = \int \bar{\partial}\bar{\theta}\partial\theta \tag{27}$$

where θ and $\bar{\theta}$ are grassman variables. This action is related to $c = -2$ conformal field theory which is known to be a logarithmic conformal field theory, that is, there can be found logarithmic terms in correlation functions of the fields inside the theory [22]. In such theories there exists pair of primary fields which transform into each other under conformal transformations.

Before considering the action (27) we would like to review a powerful method developed to investigate LCFT's, namely the nilpotent weight method. The method is primarily based on a composite field, containing the two primary fields and a nilpotent variable which acts as a part of filed's weight[24], that is, from the primary fields ϕ and ψ we construct the field $\Phi(z, \beta) = \phi(z) + \beta\,\psi(z)$ where β is a nilpotent variable. This fields acts as a primary field under conformal transformations with the weight $h + \beta$, assuming ϕ has weight h. The idea was then generalized in [25], where the nilpotent variable was taken to be product of a grassman variable and its conjugate, hence the composite field contained four different fields, two bosonic and two fermionic. This structure was observed before in $c = -2$ theory by Kausch [26]. In this language the composite is written to be

$$\Phi(z, \eta) = \phi(z) + \bar{\eta}\xi(z) + \bar{\xi}(z)\eta + \bar{\eta}\eta\psi(z) \tag{28}$$

The fields ϕ and ψ are the same as previous ones and ξ and $\bar{\xi}$ are the new primary fermionic fields. The two point correlation function of such fields can be found exploiting conformal invariance:

$$\langle \Phi(z_1, \eta_1)\Phi(z_2, \eta_2)\rangle = \frac{(\bar{\eta}_1 + \bar{\eta}_2)(\eta_1 + \eta_2)}{(z_1 - z_2)^{2h + \bar{\eta}_1\eta_1 + \bar{\eta}_2\eta_2}} \tag{29}$$

This equation could be read through expanding both sides of it, in terms of grassman variables to find correlation function of individual fields.

With this brief review, we will go back to our problem, a $c = -2$ theory with different boundary conditions. Assume that the boundary condition is Drichlet with discontinuities α_j at points x_j. We should do the same procedure as in the bosonic case. The classical solution satisfying this BC is

$$\theta_{cl} = i \sum \alpha_j \ln\left(\frac{z - x_j}{\bar{z} - x_j}\right) \tag{30}$$

Here α_j's are grassman variables in contrast with the bosonic case. The next step is to calculate the classical part of the partition function. As the solution (30) is just the same

as the bosonic one, the result is the same, too. The only difference is that we have $\bar{\theta}$ in the action and this variable's jump is $\bar{\alpha}$ rather than α. So we have

$$Z_\alpha = \prod_{j \neq k} \left(\frac{x_k - x_j}{a} \right)^{\bar{\alpha}_j \alpha_k} . \tag{31}$$

Now we can assign this partition function to correlation of boundary fields. Before doing this we have to impose a condition similar to charge neutrality in the bosonic case, that is we should assume that the sum of all the jumps has to be zero. To assure that this happens we define

$$\langle \prod_j \vartheta(\alpha_j, x_j) \rangle = Z_\alpha \times \delta(\sum_j \alpha_j) \delta(\sum_j \bar{\alpha}_j). \tag{32}$$

Here ϑ_j's are the boundary fields and could be expanded in terms of α_j's to have $\vartheta(x, \alpha) = \phi(x) + \bar{\alpha}\xi(x) + \bar{\xi}(x)\alpha + \bar{\alpha}\alpha\psi(z)$. Let's investigate the two point correlation of such fields. As the delta function of a grassman variable is itself, we'll have

$$\langle \vartheta(x_1, \alpha_1) \vartheta(x_2, \alpha_2) \rangle \propto (x_1 - x_2)^{(\bar{\alpha}_1 \alpha_2 + \bar{\alpha}_2 \alpha_1)} (\bar{\alpha}_1 + \bar{\alpha}_2)(\alpha_1 + \alpha_2) \tag{33}$$

Comparing with equation (29) and noting that sum of α_j's is vanishing, we observe that we have found the correct result. Note that the weight of this field is zero as expected in $c = -2$ theory. It may be possible to construct a non zero weight boundary field if we let the action (27) have a bosinic part. Higher correlation functions could be also derived in the same way, but now we would like to find the OPE of the fields with energy-momentum tensor. The expectation value of energy momentum tensor could be read easily through

$$\langle T(z) \rangle_\alpha = -\bar{\partial} \bar{\theta}_{cl} \partial \theta_{cl} = \sum_{j,k} \frac{\bar{\alpha}_j \alpha_k}{(x_j - x_k)}. \tag{34}$$

which leads to the OPE:

$$T(z)\vartheta(x_j, \alpha_j) = \frac{\bar{\alpha}_j \alpha_j}{(z - x_j)^2} + \frac{2}{z - x_j} \sum{}' \left(\frac{\bar{\alpha}_j \alpha_k}{x_j - x_k} + \frac{\bar{\alpha}_k \alpha_j}{x_k - x_j} \right) + \ldots, \tag{35}$$

which means that ϑ fields have weight equal to $\bar{\alpha}\alpha$, which is consistent with previous results [24, 25]. Using these boundary operators and their correlation functions, one is able to find the related SLE(κ, ρ), exploiting the method introduced by Bernard *et al.* [30].

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